MATHEMATICS

# STABILITY PROPERTIES OF FINITE MEROMORPHIC OPERATOR FUNCTIONS. II

BY

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## 3. *Reduction to the holomorphic case*

We begin with two lemmas which will be useful both here and in other sections. The proof of Lemma 3.1(b) follows immediately from Theorem IV.1.12 in [11]; the proof of the other statements are elementary and will be omitted. We say that a subspace $M$ of a Banach space $Z$ is *complemented* (in $Z$) if $M$ is closed and there exists a closed subspace $N$ of $Z$ such that $Z=M \oplus N$.

3.1 LEMMA. *Let $M$ and $N$ be subspaces of $X$ such that $N \subset M$ and* $\dim M/N<\infty$.

(a) *If $N$ is closed, then $M$ is closed.*
(b) *If $M$ is closed and if $N=R(T)$ for some bounded linear operator $T$ from a Banach space $Z$ into $X$, then $N$ is closed.*
(c) *If $M$ is complemented and $N$ is closed, then $N$ is complemented.*

3.2 LEMMA. *Let $M$ and $N$ be subspaces of $X$ such that $N \subset M$. Let $P$ be a continuous projection of $X$ onto $N$. Then*

$$M=N \oplus (I-P)M=N \oplus (M \cap N(P)).$$

*If, in addition,* $\dim M/N<\infty$, *then $M$ is complemented.*

Recall that $A$ denotes an operator function, holomorphic on a deleted neighbourhood of $\lambda_0$ and having values in $\mathscr{L}(X, Y)$. We say that $A$ is *finite meromorphic* at $\lambda_0$ if $\nu(A;\lambda_0)>-\infty$ and the coefficients in the principal part of the Laurent expansion of $A$ at $\lambda_0$ are all degenerate operators (that is, operators with finite-dimensional range spaces). Observe that any diagonal operator function at $\lambda_0$ is finite meromorphic at $\lambda_0$.

Finite meromorphic operator functions which have the extra property that the constant term in the Laurent expansion at $\lambda_0$ is a Fredholm operator have been studied by several authors (see [1], [2], [4], [6], [8], [9], [10], [12], [24] and [25]). A characterization of such operator functions in terms of the spaces $H_0$ and $K_0$ is given at the end of this section. Other subclasses are considered in [1], [2], [12] and [24]. In this section we show (among other things) how the study of finite meromorphic functions can be reduced to that of holomorphic functions. The method we develop here is based on an idea contained in [9].

3.3 PROPOSITION. *Suppose* $\nu(A;\lambda_0)>-\infty$. *Then* $A$ *is finite meromorphic at* $\lambda_0$ *if and only if* $\dim K_{-1}[A;\lambda_0]<\infty$.

PROOF. We may suppose $\nu(A;\lambda_0)=-p$, where $p>0$. Then

$$A(\lambda)=\sum_{n=-p}^{\infty}(\lambda-\lambda_0)^n A_n. \tag{3-1}$$

It follows easily from the definition of $K_{-1}[A;\lambda_0]$ that

$$K_{-1}[A;\lambda_0]\subset R(A_{-1})+\ldots+R(A_{-p}).$$

Hence if $A$ is finite meromorphic at $\lambda_0$, then $K_{-1}[A;\lambda_0]$ must be finite-dimensional. Now suppose that $\dim K_{-1}[A;\lambda_0]<\infty$. Using Proposition 1.3, we have

$$R(A_{-p})=K_{-p}[A;\lambda_0]\subset K_{-1}[A;\lambda_0].$$

Hence $A_{-p}$ is a degenerate operator. If $p>1$, we may complete the proof by finite induction. Suppose that $A_{-p},\ldots,A_{-k}$ are degenerate operators for some $k$, $1<k\leqslant p$. Let

$$W=\bigcap_{i=k}^{p}N(A_{-i}).$$

Then $W$ is a closed subspace of finite codimension in $X$. Given $w\in W$, let $\phi(\lambda)=(\lambda-\lambda_0)^{k-1}w$. Then $\phi\in\mathscr{H}(\lambda_0,X)$, $\nu(\phi;\lambda_0)\geqslant-(-k+1)$ and $A(\lambda)\phi(\lambda)\to\to A_{-k+1}w$. Thus $A_{-k+1}w$ belongs to $K_{-k+1}$, which proves that

$$A_{-k+1}(W)\subset K_{-k+1}[A;\lambda_0]\subset K_{-1}[A;\lambda_0].$$

Since $\dim X/W<\infty$, it follows that $A_{-k+1}$ is a degenerate operator. By finite induction, this implies that $A_{-p},\ldots,A_{-1}$ are all degenerate operators.

In the rest of this section we assume that $A$ is finite meromorphic at $\lambda_0$ and $\nu(A;\lambda_0)\geqslant-p$, for some positive integer $p$. Thus $A$ has the form (3–1) with $A_{-1},\ldots,A_{-p}$ degenerate. We shall construct two holomorphic operator functions $S$ and $T$ that will be useful in the study of properties of $A$.

Let $Q_1$ be a continuous projection of $Y$ such that $R(Q_1)$ is finite-dimensional and

$$Q_0A_{-i}=0,\ i=1,\ldots,p, \tag{3-2}$$

where $Q_0=I_Y-Q_1$. For instance, let $Q_1$ be a continuous projection of $Y$ onto the finite-dimensional space $R(A_{-1})+\ldots+R(A_{-p})$. Define $B$: $\mathbf{C}\to\to\mathscr{L}(Y)$ by

$$B(\lambda)=Q_0+(\lambda-\lambda_0)^{p+1}Q_1. \tag{3-3}$$

Let $\Delta$ be a deleted neighbourhood of $\lambda_0$ such that $A$ is holomorphic on $\Delta$, and define $S$ on $\Delta \cup \{\lambda_0\}$ by

$$S(\lambda) = \begin{cases} B(\lambda)A(\lambda), & \lambda \in \Delta, \\ Q_0 A_0, & \lambda = \lambda_0. \end{cases} \tag{3–4}$$

It is evident that $S \in \mathscr{H}(\lambda_0, \mathscr{L}(X, Y))$. Furthermore, it follows from (3–2) that $S$ is holomorphic at $\lambda_0$. Observe that $B(\lambda)$ is bijective when $\lambda \neq \lambda_0$, and

$$B(\lambda)^{-1} = Q_0 + (\lambda - \lambda_0)^{-p-1} Q_1.$$

Hence

$$A(\lambda) = B(\lambda)^{-1} S(\lambda), \ \lambda \in \Delta. \tag{3–5}$$

For brevity, we write $B^{-1}$ for the function $\lambda \to B(\lambda)^{-1}$. The following lemma follows easily from (3–4), (3–5) and the facts that $\nu(B; \lambda_0) \geqslant 0$ and $\nu(B^{-1}; \lambda_0) \geqslant -(p+1)$.

3.4 Lemma. *Let $A$ be finite meromorphic at $\lambda_0$, and let $S$ be as above. Then, for $m \in \mathbb{Z}$,*

$$H_{m+p+1}[S; \lambda_0] \subset H_m[A; \lambda_0] \subset H_m[S; \lambda_0],$$

*and, for each $\lambda$ in $\Delta \cup \{\lambda_0\}$,*

$$H[S; \lambda] = H[A; \lambda], \ H_c[S; \lambda] = H_c[A; \lambda].$$

The function $S$ is holomorphic at $\lambda_0$, and hence

$$H_0[S; \lambda_0] = N(S(\lambda_0)) = N(Q_0 A_0).$$

It is easily verified that

$$\frac{N(Q_0 A_0)}{N(A_0)} \simeq R(A_0) \cap N(Q_0).$$

Since $N(Q_0) = R(Q_1)$ is finite-dimensional, it follows that

$$\dim \frac{H_0[S; \lambda_0]}{N(A_0)} < \infty. \tag{3–6}$$

If we let

$$W = \bigcap_{i=1}^{p} N(A_{-i}), \tag{3–7}$$

then it is easily verified that

$$N(A_0) \cap W \subset H_0[A; \lambda_0] \subset H_0[S; \lambda_0]. \tag{3–8}$$

Note that $W$ has finite codimension in $X$, because $A_{-1}, \ldots, A_{-p}$ are degenerate operators. Using (3–6), we have

$$\dim \frac{H_0[S; \lambda_0]}{N(A_0) \cap W} \leqslant \dim \frac{H_0[S; \lambda_0]}{N(A_0)} + \dim \frac{X}{W} < \infty. \tag{3–9}$$

Consequently, from (3–8),

$$\dim \frac{H_0[S;\lambda_0]}{H_0[A;\lambda_0]} < \infty. \tag{3–10}$$

We shall use these facts in some of the results which follow.

3.5 PROPOSITION. *Let $A$ be finite meromorphic at $\lambda_0$, and let $S$ be as above. Then*

(a) *$R(S(\lambda_0))$ is closed if and only if $R(A_0)$ is closed,*

(b) *$k(S;\lambda_0)<\infty$ if and only if $k(A;\lambda_0)<\infty$.*

PROOF. Let $M=N(Q_0)+R(A_0)$. Since $N(Q_0)$ is finite-dimensional, Lemma 3.1 shows that $R(A_0)$ is closed if and only if $M$ is closed. Since $Q_0$ is a projection with $N(Q_0)\subset M$, it follows that

$$M \cap R(Q_0)=Q_0 M \subset M.$$

Observe that $R(S(\lambda_0))=R(Q_0 A_0)=Q_0 M$. Thus

$$\dim \frac{M}{R(S(\lambda_0))} = \dim \frac{M}{M\cap R(Q_0)} \leqslant \dim \frac{X}{R(Q_0)} = \dim N(Q_0) < \infty.$$

But then we can apply Lemma 3.1 again to show that $M$ is closed if and only if $R(S(\lambda_0))$ is closed. This proves (a).

From Lemma 3.4, we have

$$H[S;\lambda_0]=H[A;\lambda_0]\subset H_0[A;\lambda_0]\subset H_0[S;\lambda_0].$$

This implies that (cf. (1–5))

$$k(S;\lambda_0)=k(A;\lambda_0)+\dim \frac{H_0[S;\lambda_0]}{H_0[A;\lambda_0]}.$$

Assertion (b) now follows from (3–10).

Let $P_0$ be a continuous projection of $X$ such that

$$A_{-i}P_0=0,\ i=1,\ldots,p, \tag{3–11}$$

and such that the range of the projection $P_1$ given by $P_1=I_X-P_0$ is finite-dimensional. For instance, let $P_0$ be a continuous projection of $X$ onto the space $W$ described in (3–7). Define $C\colon \mathbf{C}\to \mathscr{L}(X)$ by

$$C(\lambda)=P_0+(\lambda-\lambda_0)^{p+1}P_1. \tag{3–12}$$

With $\Delta$ as in (3–4), define $T$ on $\Delta \cup \{\lambda_0\}$ by

$$T(\lambda)=\begin{cases} A(\lambda)\,C(\lambda), & \lambda\in\Delta,\\ A_0P_0, & \lambda=\lambda_0.\end{cases} \tag{3–13}$$

Clearly $T \in \mathscr{H}(\lambda_0, \mathscr{L}(X, Y))$. Also, $T$ is holomorphic at $\lambda_0$, because of (3–11). Observe that $C(\lambda)$ is bijective when $\lambda \neq \lambda_0$, and

$$A(\lambda) = T(\lambda)\, C(\lambda)^{-1},\ \lambda \in \Delta.$$

The following lemma can be proved in much the same way as Lemma 3.4.

3.6 LEMMA. *Let $A$ be finite meromorphic at $\lambda_0$, and let $T$ be as above. Then, for $m \in \mathbb{Z}$,*

$$K_{m-p-1}[A\,;\lambda_0] \subset K_m[T\,;\lambda_0] \subset K_m[A\,;\lambda_0],$$

*and, for each $\lambda$ in $\Delta \cup \{\lambda_0\}$,*

$$K[T\,;\lambda] = K[A\,;\lambda],\ K_c[T\,;\lambda] = K_c[A\,;\lambda].$$

Observe now that

$$K_0[T\,;\lambda_0] = R(T(\lambda_0)) = R(A_0 P_0) \subset R(A_0).$$

Also,

$$R(A_0) = R(A_0 P_0) + R(A_0 P_1) = R(T(\lambda_0)) + R(A_0 P_1). \tag{3–14}$$

Since $P_1$ has finite-dimensional range, $A_0 P_1$ must be a degenerate operator. It follows that

$$\dim \frac{R(A_0)}{R(T(\lambda_0))} < \infty. \tag{3–15}$$

It is easily verified that

$$K_0[A\,;\lambda_0] \subset R(A_0) + M,$$

where $M = R(A_{-1}) + \ldots + R(A_{-p})$. Note that $\dim M < \infty$. Using (3–14), we have

$$K_0[A\,;\lambda_0] \subset R(T(\lambda_0)) + R(A_0 P_1) + M.$$

Since $R(T(\lambda_0)) = K_0[T\,;\lambda_0] \subset K_0[A\,;\lambda_0]$, by Lemma 3.6, we conclude that

$$\dim \frac{K_0[A\,;\lambda_0]}{K_0[T\,;\lambda_0]} = \dim \frac{K_0[A\,;\lambda_0]}{R(T(\lambda_0))} < \infty. \tag{3–16}$$

3.7 PROPOSITION. *Let $A$ be finite meromorphic at $\lambda_0$, and let $T$ be as above. Then*

(a) *$R(T(\lambda_0))$ is closed if and only if $R(A_0)$ is closed,*
(b) *$k(T\,;\lambda_0) < \infty$ if and only if $k(A\,;\lambda_0) < \infty$.*

PROOF. Assertion (a) follows from (3–15) and Lemma 3.1. From Lemma 3.6, we have

$$K_0[T\,;\lambda_0] \subset K_0[A\,;\lambda_0] \subset K[A\,;\lambda_0] = K[T\,;\lambda_0].$$

This implies that (cf. (1–5))

$$k(T;\lambda_0)=k(A;\lambda_0)+\dim\frac{K_0[A;\lambda_0]}{K_0[T;\lambda_0]}.$$

Assertion (b) now follows from (3–16).

3.8 PROPOSITION. *Let $A$ be finite meromorphic at $\lambda_0$. Then*
(a) *$H_0[A;\lambda_0]$ is closed,*
(b) *$\dim H_0[A;\lambda_0]<\infty$ if and only if $\dim N(A_0)<\infty$,*
(c) *$H_0[A;\lambda_0]$ is complemented if and only if $N(A_0)$ is complemented,*
(d) *$K_0[A;\lambda_0]$ is closed if and only if $R(A_0)$ is closed,*
(e) *$\operatorname{codim} K_0[A;\lambda_0]<\infty$ if and only if $\operatorname{codim} R(A_0)<\infty$,*
(f) *$K_0[A;\lambda_0]$ is complemented if and only if $R(A_0)$ is complemented.*

PROOF. If $W$ is given by (3–7), then (3–8) and (3–9) imply that

$$\dim\frac{H_0[A;\lambda_0]}{N(A_0)\cap W}<\infty. \tag{3–17}$$

Since $N(A_0)\cap W$ is obviously closed, (a) follows from Lemma 3.1. Now $\dim X/W<\infty$, and so

$$\dim\frac{N(A_0)}{N(A_0)\cap W}<\infty. \tag{3–18}$$

Statement (b) follows from (3–17) and (3–18). Using Lemmas 3.1 and 3.2 with (3–17) and (3–18), one obtains (c). Let $T$ be as above. Then (d) and (e) follow from (3–15), (3–16) and Lemma 3.1. To prove (f), observe that if either $K_0[A;\lambda_0]$ or $R(A_0)$ is complemented, then they are both closed, by part (d). So $R(T(\lambda_0))$ is closed in either case, by Proposition 3.7. Statement (f) now follows from (3–15), (3–16) and Lemmas 3.1 and 3.2.

3.9 COROLLARY. *Let $A$ be finite meromorphic at $\lambda_0$. Then $A_0$ is a Fredholm operator if and only if*

$$\dim H_0[A;\lambda_0]<\infty \text{ and } \operatorname{codim} K_0[A;\lambda_0]<\infty.$$

*In this case, $R(A_0)$ is closed and $k(A;\lambda_0)<\infty$.*

## 4. *Reduction to the linear case*

In this section we shall describe in detail a method which allows us to reduce the study of a holomorphic operator function to that of a function of the form $T+\lambda S$. Linearization techniques have often been used in the study of operator polynomials and, less frequently, in the study of holomorphic operator functions (see [2] and the references given there). The method described here is a further elaboration of the one used by H. BART in [2].

Throughout this section $A$ will be a function with values in $\mathscr{L}(X, Y)$, and $A$ will be holomorphic on an open neighbourhood of a point $\lambda_0$ in $\mathbf{C}$. For the sake of simplicity we assume that $\lambda_0=0$. With $A$ we shall associate a linear function $\boldsymbol{L}(\lambda)=\boldsymbol{T}+\lambda\boldsymbol{S}$ such that $\boldsymbol{T}$ and $\boldsymbol{S}$ are bounded linear operators acting between certain Banach spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$.

We begin with the construction of the spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. The space $\boldsymbol{X}$ is the linear subspace of the product space

$$\prod=\prod_{k=0}^{\infty} X_k,\ X_k=X,\ k=0, 1, \ldots, \tag{4–1}$$

consisting of all sequences $\boldsymbol{x}=\{x_k\}_{k=0}^{\infty}\in\prod$ which have the property that

$$\sup\{\|x_k\|: k=0, 1, \ldots\}<\infty.$$

Let $\boldsymbol{X}$ have the norm given by

$$\|\boldsymbol{x}\|=\sup_k \|x_k\|,\ \boldsymbol{x}\in\boldsymbol{X}.$$

Then $\boldsymbol{X}$ is a complex Banach space. The complex Banach space $\boldsymbol{Y}$ is defined in the same way as $\boldsymbol{X}$ except that in (4–1) we take $X_0=Y$ instead of $X_0=X$.

Next we define the operators $\boldsymbol{T}$ and $\boldsymbol{S}$. Let $A_n$ denote the $n^{\text{th}}$ coefficient of the Taylor expansion of $A$ at 0. Choose $r>0$ such that $A$ is holomorphic on an open neighbourhood of the set $\{\lambda: |\lambda|\leq r\}$, and let $\Delta_r$ be the open disc $\{\lambda: |\lambda|<r\}$. Then

$$\sum_{n=0}^{\infty} r^n\|A_n\|<\infty. \tag{4–2}$$

The operators $\boldsymbol{T}$ and $\boldsymbol{S}$ are defined on $\boldsymbol{X}$ by the following formulas:

$$\boldsymbol{T}\boldsymbol{x}=(A_0 x_0, x_1, x_2, \ldots)$$

and

$$\boldsymbol{S}\boldsymbol{x}=\left(\sum_{n=1}^{\infty} r^{n-1}A_n x_{n-1},\ -\frac{1}{r}x_0,\ -\frac{1}{r}x_1,\ \ldots\right).$$

From (4–2) it follows that the series appearing in the definition of $\boldsymbol{S}$ converges. It is not difficult to show that $\boldsymbol{T}$ and $\boldsymbol{S}$ are well-defined bounded linear operators from $\boldsymbol{X}$ into $\boldsymbol{Y}$. In the remainder of this section $\boldsymbol{L}$ will be the operator function defined on $\mathbf{C}$ by

$$\boldsymbol{L}(\lambda)=\boldsymbol{T}+\lambda\boldsymbol{S}.$$

In order to describe certain useful relations between $A$ and $\boldsymbol{L}$, we need some auxiliary operator functions.

Define $\Psi$ from $\Delta_r$ into $\mathscr{L}(X, \boldsymbol{X})$ by

$$\Psi(\lambda)x=\left(x, \frac{\lambda}{r}x, \left(\frac{\lambda}{r}\right)^2 x, \ldots\right).$$

Since $\Psi(\lambda)$ can be written as a power series which converges in $\mathscr{L}(X, \mathbf{X})$, the operator function $\Psi$ is holomorphic on $\Delta_r$. Note that each operator $\Psi(\lambda)$ is an isometry from $X$ into $\mathbf{X}$.

Next, let $\Xi$ be the canonical imbedding of $Y$ into $\mathbf{Y}$, that is,

$$\Xi(y) = (y, 0, 0, \dots).$$

A simple lemma follows immediately from these definitions.

4.1 Lemma. *For $\lambda \in \Delta_r$, we have*

$$\mathbf{L}(\lambda)\Psi(\lambda) = \Xi A(\lambda).$$

Now take $\lambda$ in $\Delta_r$, $\mathbf{x}$ in $\mathbf{X}$ and $\mathbf{z}$ in $\mathbf{Y}$, and suppose that

$$\mathbf{z} = (\mathbf{T} + \lambda\mathbf{S})\mathbf{x}.$$

Then we have

$$z_0 = A_0 x_0 + \sum_{n=1}^{\infty} \lambda r^{n-1} A_n x_{n-1} \tag{4–3}$$

and

$$z_k = x_k - \left(\frac{\lambda}{r}\right) x_{k-1}, \quad k = 1, 2, \dots . \tag{4–4}$$

From (4–4) it is easy to deduce that

$$x_k = \left(\frac{\lambda}{r}\right)^k x_0 + \sum_{j=0}^{k-1} \left(\frac{\lambda}{r}\right)^j z_{k-j}, \quad k = 1, 2, \dots . \tag{4–5}$$

For $k = 1, 2, \dots$, define operator functions $\Phi_k$ from $\Delta_r$ into $\mathscr{L}(\mathbf{Y}, X)$ by

$$\Phi_k(\lambda)\mathbf{z} = \sum_{j=0}^{k-1} \left(\frac{\lambda}{r}\right)^j z_{k-j}, \quad \mathbf{z} \in \mathbf{Y}.$$

Since $z_1, z_2, \dots$, are in $X$, the vector $\Phi_k(\lambda)\mathbf{z}$ is in $X$. We may write (4–5) in the form

$$x_k = \left(\frac{\lambda}{r}\right)^k x_0 + \Phi_k(\lambda)\,\mathbf{z}, \quad k = 1, 2, \dots . \tag{4–6}$$

Combining this with (4–3) (and using (4–2)), we have

$$z_0 = A(\lambda)\, x_0 + \sum_{n=2}^{\infty} \lambda r^{n-1} A_n \Phi_{n-1}(\lambda)\,\mathbf{z}. \tag{4–7}$$

Define an operator function $\Phi$ from $\Delta_r$ into $\mathscr{L}(\mathbf{Y}, Y)$ by

$$\Phi(\lambda)\,\mathbf{z} = z_0 - \sum_{n=1}^{\infty} \lambda r^n A_{n+1} \Phi_n(\lambda)\,\mathbf{z}, \ \ \mathbf{z} \in \mathbf{Y}. \tag{4–8}$$

Formula (4–2) together with the fact that

$$\|\Phi_n(\lambda)\| \leqslant \frac{r}{r - |\lambda|}, \ \lambda \in \Delta_r, \quad n = 1, 2, \dots , \tag{4–9}$$

implies that $\Phi(\lambda)$ is a well-defined bounded linear operator from $\mathbf{Y}$ into $Y$. Furthermore, using (4–9) again, the series

$$\sum_{n=1}^{\infty} \lambda r^n A_{n+1} \Phi_n(\lambda)$$

converges uniformly (in the norm of $\mathscr{L}(\mathbf{Y}, Y)$) on each compact subset of $\Delta_r$. Hence the operator function $\Phi$ is holomorphic on $\Delta_r$. Using $\Phi(\lambda)$ in (4–7), we see that

$$A(\lambda)\, x_0 = \Phi(\lambda)\, \mathbf{z}. \tag{4–10}$$

Finally, for $n = 0, 1, 2, \ldots$, define operators $\Theta_n$ in $\mathscr{L}(\mathbf{X}, X)$ by

$$\Theta_n \mathbf{x} = x_n, \quad \mathbf{x} \in \mathbf{X}.$$

It is convenient to summarize the results of (4–6) and (4–10) in the following lemma.

4.2 Lemma. *For $\lambda \in \Delta_r$, we have*

(a) $\Theta_n = (\lambda/r)^n\, \Theta_0 + \Phi_n(\lambda)\, \mathbf{L}(\lambda)$, $n = 1, 2, \ldots$,

(b) $A(\lambda)\, \Theta_0 = \Phi(\lambda)\, \mathbf{L}(\lambda)$.

It is interesting to observe that for $\lambda \in \Delta_r$ the operator $\Phi(\lambda)$ is surjective. This follows from the following equation:

$$\Phi(\lambda)\, \Xi = I_Y, \quad \lambda \in \Delta_r. \tag{4–11}$$

For $\Psi(\lambda)$ we have a similar result:

$$\Theta_0\, \Psi(\lambda) = I_X, \quad \lambda \in \Delta_r. \tag{4–12}$$

If we use (4–12) in Lemma 4.2, we obtain the following result.

4.3 Proposition. *For $\lambda \in \Delta_r$, we have*

$$A(\lambda) = \Phi(\lambda)\, \mathbf{L}(\lambda)\, \Psi(\lambda).$$

4.4 Lemma. *For $\lambda \in \Delta_r$, we have*

(a) $N(\mathbf{L}(\lambda)) \subset R(\Psi(\lambda))$,

(b) $R(\mathbf{L}(\lambda)) \supset N(\Phi(\lambda))$.

Proof. (a) Take $\mathbf{x}$ in $N(\mathbf{L}(\lambda))$. Lemma 4.2(a) shows that, for $k = 1, 2, \ldots$,

$$x_k = \Theta_k \mathbf{x} = \left(\frac{\lambda}{r}\right)^k \Theta_0 \mathbf{x} = \left(\frac{\lambda}{r}\right)^k x_0,$$

and thus $\mathbf{x} = \Psi(\lambda)\, x_0 \in R(\Psi(\lambda))$.

(b) Take $\mathbf{v}$ in $N(\Phi(\lambda))$. Define $\mathbf{x}$ in $\mathbf{X}$ by setting

$$x_k = \begin{cases} 0 & \text{for } k = 0, \\ \Phi_k(\lambda)\, \mathbf{v} & \text{for } k \neq 0. \end{cases}$$

Formula (4–9) shows that $\mathbf{x} \in \mathbf{X}$. Put $\mathbf{z} = (\mathbf{T} + \lambda \mathbf{S})\mathbf{x}$. Then, by (4–3) and the definition (4–8) of $\Phi(\lambda)$,

$$z_0 = \sum_{n=2}^{\infty} \lambda r^{n-1} A_n \Phi_{n-1}(\lambda)\, \mathbf{v} = v_0 - \Phi(\lambda)\, \mathbf{v} = v_0.$$

Furthermore, from (4–4),

$$z_k = x_k - \left(\frac{\lambda}{r}\right) x_{k-1}, \quad k = 1, 2, \dots .$$

Hence $z_1 = x_1 = \Phi_1(\lambda)\mathbf{v} = v_1$, and, for $k = 2, 3, \dots$,

$$z_k = \Phi_k(\lambda)\, \mathbf{v} - \left(\frac{\lambda}{r}\right) \Phi_{k-1}(\lambda)\, \mathbf{v} = v_k.$$

This shows that $\mathbf{z} = \mathbf{v}$, and hence $\mathbf{v} \in R(\mathbf{L}(\lambda))$.

4.5 PROPOSITION. *For $\lambda \in \Delta_r$, we have*

(a) $H_m[\mathbf{L}; \lambda] = \Psi(\lambda)\, H_m[A; \lambda]$, $m = 0, 1, 2, \dots$,

(b) $H[\mathbf{L}; \lambda] = \Psi(\lambda)\, H[A; \lambda]$,

(c) $H_c(\mathbf{L}; \lambda] = \Psi(\lambda)\, H_c[A; \lambda]$.

PROOF. (a) Take a fixed $\lambda_0$ in $\Delta_r$, and let $m$ be a fixed nonnegative integer. Given $x$ in $H_m[A; \lambda_0]$, there exists $\phi \in \mathscr{H}(\lambda_0, X)$ such that

$$\phi(\lambda) \to x, \ \nu(A\phi; \lambda_0) \geqslant m+1.$$

Define $\psi$ on a deleted neighbourhood of $\lambda_0$ by

$$\psi(\lambda) = \Psi(\lambda)\, \phi(\lambda). \tag{4–13}$$

Since $\Psi$ is holomorphic on $\Delta_r$, the function $\psi$ belongs to $\mathscr{H}(\lambda_0, \mathbf{X})$. Further,

$$\psi(\lambda) \to \Psi(\lambda_0)\, x.$$

By Lemma 4.1,

$$\mathbf{L}(\lambda)\, \psi(\lambda) = \Xi A(\lambda)\, \phi(\lambda)$$

for $\lambda$ in some deleted neighbourhood of $\lambda_0$, and so

$$\nu(\mathbf{L}\psi; \lambda_0) \geqslant \nu(A\phi; \lambda_0) \geqslant m+1.$$

Thus $\Psi(\lambda_0)x \in H_m[\mathbf{L}; \lambda_0]$, and it follows that

$$\Psi(\lambda_0)\, H_m[A; \lambda_0] \subset H_m[\mathbf{L}; \lambda_0].$$

To prove the reverse inclusion, take $\mathbf{w}$ in $H_m[\mathbf{L}; \lambda_0]$. Then there exists $\psi \in \mathscr{H}(\lambda_0, \mathbf{X})$ such that

$$\psi(\lambda) \to \mathbf{w}, \ \nu(\mathbf{L}\psi; \lambda_0) \geqslant m+1.$$

Define $\phi$ on a deleted neighbourhood of $\lambda_0$ by

$$\phi(\lambda) = \Theta_0\, \psi(\lambda). \tag{4–14}$$

Then $\phi \in \mathscr{H}(\lambda_0, X)$ and

$$\phi(\lambda) \to \Theta_0 \mathbf{w}.$$

By Lemma 4.2(b),

$$A(\lambda)\,\phi(\lambda) = \Phi(\lambda)\,\mathbf{L}(\lambda)\,\psi(\lambda)$$

for $\lambda$ in some deleted neighbourhood of $\lambda_0$. Since $\Phi$ is holomorphic on $\Delta_r$, it follows that

$$\nu(A\phi;\lambda_0) \geqslant \nu(\mathbf{L}\psi;\lambda_0) \geqslant m+1.$$

Hence $\Theta_0\mathbf{w} \in H_m[A;\lambda_0]$. Now it follows from (4–12) that $\psi(\lambda_0)\Theta_0$ is the identity operator on $R(\Psi(\lambda_0))$. Notice that $\mathbf{w} \in N(\mathbf{L}(\lambda_0)) \subset R(\Psi(\lambda_0))$, by Lemma 4.4(a). Thus we have

$$\mathbf{w} = \Psi(\lambda_0)[\Theta_0\,\mathbf{w}] \in \Psi(\lambda_0)\,H_m[A;\lambda_0].$$

Combining this with the result of the first paragraph of this proof, we obtain (a).

(b) This follows from (a) by taking intersections and using the fact that $\Psi(\lambda)$ is injective.

(c) The proof of (c) follows the same pattern as that of (a) and is therefore omitted.

4.6 Proposition. *For $\lambda \in \Delta_r$, we have*

(a) $K_m[\mathbf{L};\lambda] = \Phi(\lambda)^{-1}K_m[A;\lambda]$, $m = 0, 1, 2, \ldots,$

(b) $K[\mathbf{L};\lambda] = \Phi(\lambda)^{-1}K[A;\lambda]$,

(c) $K_c[\mathbf{L};\lambda] = \Phi(\lambda)^{-1}K_c[A;\lambda]$.

Proof. (a) Take a fixed $\lambda_0$ in $\Delta_r$, and let $m$ be a fixed nonnegative integer. We begin by observing that

$$N(\Phi(\lambda_0)) \subset K_m[\mathbf{L};\lambda_0]. \tag{4–15}$$

This follows from Lemma 4.4(b) and the fact that

$$R(\mathbf{L}(\lambda_0)) = K_0[\mathbf{L};\lambda_0] \subset K_m[\mathbf{L};\lambda_0].$$

Given $y$ in $K_m[A;\lambda_0]$, there exists $\phi \in \mathscr{H}(\lambda_0, X)$ such that

$$\nu(\phi;\lambda_0) \geqslant -m,\ (A\phi)(\lambda) \to y.$$

Define $\psi$ by (4–13). Then $\psi \in \mathscr{H}(\lambda_0, \mathbf{X})$ and, as $\Psi$ is holomorphic on $\Delta_r$,

$$\nu(\psi;\lambda_0) \geqslant \nu(\phi;\lambda_0) \geqslant -m.$$

Further, by Lemma 4.1,

$$\mathbf{L}(\lambda)\,\psi(\lambda) = \Xi(A(\lambda)\,\phi(\lambda)) \to \Xi(y).$$

This shows that $\Xi(y) \in K_m[\mathbf{L};\lambda_0]$. Now, by (4–11),

$$y = \Phi(\lambda_0)\,\Xi(y).$$

Thus $y \in \Phi(\lambda_0) K_m[\mathbf{L}; \lambda_0]$, and it follows that

$$K_m[A; \lambda_0] \subset \Phi(\lambda_0) K_m[\mathbf{L}; \lambda_0].$$

By combining this with (4–15), we see that

$$\Phi(\lambda_0)^{-1} K_m[A; \lambda_0] \subset K_m[\mathbf{L}; \lambda_0]. \tag{4–16}$$

To prove the reverse inclusion, take $\mathbf{z}$ in $K_m[\mathbf{L}; \lambda_0]$. Then there exists $\psi \in \mathscr{H}(\lambda_0, \mathbf{X})$ such that

$$\nu(\psi; \lambda_0) \geqslant -m, \quad \mathbf{L}(\lambda)\, \psi(\lambda) \to \mathbf{z}.$$

Define $\phi$ by (4–14). Then $\phi \in \mathscr{H}(\lambda_0, X)$ and

$$\nu(\phi; \lambda_0) \geqslant \nu(\psi; \lambda_0) \geqslant -m.$$

Further, by Lemma 4.2(b),

$$A(\lambda)\, \phi(\lambda) = \Phi(\lambda)\, \mathbf{L}(\lambda)\, \psi(\lambda) \to \Phi(\lambda_0)\, \mathbf{z}.$$

This shows that $\Phi(\lambda_0)\mathbf{z} \in K_m[A; \lambda_0]$. Hence

$$\Phi(\lambda_0) K_m[\mathbf{L}; \lambda_0] \subset K_m[A; \lambda_0].$$

Together with (4–16), this proves (a).

(b) This follows from (a) by taking unions.

(c) The proof of (c) follows the same pattern as that of (a) and is therefore omitted.

4.7 PROPOSITION. *For $\lambda \in \Delta_r$, we have*

(a) $\dim N(A(\lambda)) = \dim N(\mathbf{L}(\lambda))$,

(b) $\operatorname{codim} R(A(\lambda)) = \operatorname{codim} R(\mathbf{L}(\lambda))$,

(c) *$A(\lambda)$ has closed range if and only if $\mathbf{L}(\lambda)$ has closed range,*

(d) $k(A; \lambda) = k(\mathbf{L}; \lambda)$.

PROOF. (a) Take a fixed $\lambda_0$ in $\Delta_r$. Since $A$ and $\mathbf{L}$ are both holomorphic at $\lambda_0$, we have

$$N(A(\lambda_0)) = H_0[A; \lambda_0], \quad N(\mathbf{L}(\lambda_0)) = H_0[\mathbf{L}; \lambda_0].$$

Hence (a) is an immediate consequence of Proposition 4.5(a) and the injectivity of the operator $\Psi(\lambda_0)$.

(b) As in (a), we have

$$R(A(\lambda_0)) = K_0[A; \lambda_0], \quad R(\mathbf{L}(\lambda_0)) = K_0[\mathbf{L}; \lambda_0],$$

and thus Proposition 4.6(a) gives

$$R(\mathbf{L}(\lambda_0)) = \Phi(\lambda_0)^{-1} R(A(\lambda_0)). \tag{4–17}$$

Since $\Phi(\lambda_0)$ is surjective (cf. (4–11)), this implies (b).

(c) Using the continuity of the operator $\Phi(\lambda_0)$, it follows from (4–17) that $R(\mathbf{L}(\lambda_0))$ is closed whenever $R(A(\lambda_0))$ is closed. The converse statement follows from the observation that

$$\Xi^{-1} R(\mathbf{L}(\lambda_0)) = [\Phi(\lambda_0)\, \Xi]^{-1} R(A(\lambda_0)) = R(A(\lambda_0)).$$

(d) This follows from the definition of the stability number, Proposition 4.5 and the fact that $\Psi(\lambda_0)$ is injective.

For $\lambda = 0$, Propositions 4.5(a) and 4.6(a) have been proved earlier by H. Bart (cf. formulas (12) and (18) in Section III.4 of [2]). The proofs here are less complicated, because we have defined the $H_m$ and $K_m$ spaces in terms of functions instead of sequences of vectors. Proposition 4.7, except for (d), is Theorem III.4.4 in [2].

*(To be continued)*